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New classes of perfectly orderable graphs

J.-L. Fouquet[a], F. Roussel[b], P. Rubio[b], H. Thuillier[b,*]

[a] *IUT Mesures Physiques, Université du Maine, Avenue O. Messiaen 72017 Le Mans Cedex, France*
[b] *Université d'Orléans, L.I.F.O., Bat IIIA, Rue Leopard de Vinci, B.P. 6759, F-45067 Orléans Cedex 2, France*

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Abstract

This paper generalizes previous works on perfectly orderable graphs by Olariu (Discrete Math. 113 (1992) 143) and by Hoàng et al. (Discrete Math. 102 (1992) 67). Chvátal defined a graph to be perfectly orderable (V. Chvátal, in: C. Berge, V. Chvátal (Eds.), Topics on Perfect Graphs, Annals of Discrete Mathematics, Vol. 21, North-Holland, Amsterdam, 1984, pp. 63–65) if there exists a linear order $<$ on its set of vertices such that no induced path $abcd$ with edges ab, bc, cd has both $a<b$ and $d<c$. Given a graph G and a vertex v in G such that $G-v$ is perfectly orderable, we set some conditions on v for which we deduce that G is perfectly orderable. Our method allows to construct a new class of such graphs, recognizable in polynomial time, containing quasi-brittle graphs, charming graphs and some other classes of perfectly orderable graphs. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

A natural way to colour the vertices of a graph $G=(V,E)$ ordered in a sequence $v_1<v_2<\cdots<v_n$ by the set of 'colours' $\{1,2,3,\ldots\}$ consists of scanning the sequence from v_1 to v_n and assigning to each v_j the smallest integer assigned to none of its neighbours v_i such that $i<j$. This way of colouring the vertices of G is called *the greedy algorithm*. Certainly, this way may use a number of colours greater than the chromatic number of G. A graph G is said to be *perfectly orderable* [2] if there exists a linear order $<$ such that no induced P_4 $abcd$ in G has both $a<b$ and $d<c$ (such a P_4 is called an *obstruction* in $(G;<)$). Obviously, such an order, called *perfect order*, is also a perfect order on every $V'\subseteq V$, therefore the family of perfectly orderable graphs is hereditary. Recall that C. Berge defined a graph G to be *perfect* if the vertices of every induced subgraph H of G can be coloured with $\omega(H)$ colours, where $\omega(H)$ is the

* Corresponding author.
E-mail address: ht@lifo.univ-orleans.fr (H. Thuillier).

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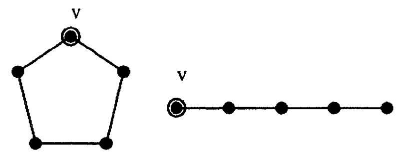

Fig. 1.

maximum clique size in H. It can be easily seen that a graph G is perfect if and only if every induced subgraph H contains a stable set intersecting every maximum clique in H. Berge and Duchet [1] defined a graph G to be *strongly perfect* if every induced subgraph H of G contains a stable set that intersects every maximal clique in H (as usual, 'maximal' is meant with respect to set-inclusion). Chvátal [2] has shown that perfectly orderable graphs are strongly perfect and that the greedy algorithm applied to a perfect order gives an optimal colouring.

For any integer $k \geqslant 3$, an induced path (respectively cycle) on k vertices will be denoted by P_k (respectively C_k). An induced subgraph of a graph G isomorphic to a P_k (respectively C_k) is simply said to be *a* P_k *in* G (resp. *a* C_k in G). A vertex v in a graph G is said to be *semi-simplicial* [6] if v is midpoint of no P_4 in G. A vertex v in a graph G is said to be *charming* [8] if v is not end-vertex in a P_5 in G, is not end-vertex in a P_5 in $\bar{G}$ and does not lie on a C_5 in G.

A *rooted graph* is a pair $H=(F,u)$ where F denotes a graph and u a vertex in F. Let $\mathscr{F}$ be a family of rooted graphs. A vertex v in a graph G is said to be $\mathscr{F}$-free if there is no induced subgraph F of G such that $H=(F,v)$ is isomorphic to a rooted graph of $\mathscr{F}$. For example, a semi-simplicial vertex is $\mathscr{F}$-free for the family $\mathscr{F}=\{(P_4,u)\}$ where u denotes a midpoint of P_4, a charming vertex is $\mathscr{F}$-free for the family $\mathscr{F}$ described in Fig. 1.

Hoàng et al. [8] call *charming* any graph in which every induced subgraph has a charming vertex, and prove that every charming graph is perfectly orderable. A natural idea is to consider the following related notion: a vertex v in a graph G is said to be *nice* if v is not internal vertex in a P_5 in G, is not internal vertex in a P_5 in $\bar{G}$ and does not lie on a C_5 in G. A nice vertex is $\mathscr{F}$-free for the family $\mathscr{F}$ described in Fig. 2. We call *nice* any graph in which every induced subgraph has a nice vertex, and we will prove that every nice graph is perfectly orderable (see below). In fact, this result is a corollary of Theorem 3 in Section 5.

An edge ab in a graph G is called a *symmetric wing* if there exist vertices c,d,p,q such that both $abcd$ and $baqp$ are induced P_4's in G. A vertex v in G is said to be *special* [10] if v is incident with no symmetric wing in G and $\bar{G}$.

Lemma 1. *An edge ab in a graph G is a symmetric wing if and only if there is an induced subgraph of G depicted in Fig.* 3 *such that ab is the edge whose ends are circled.*

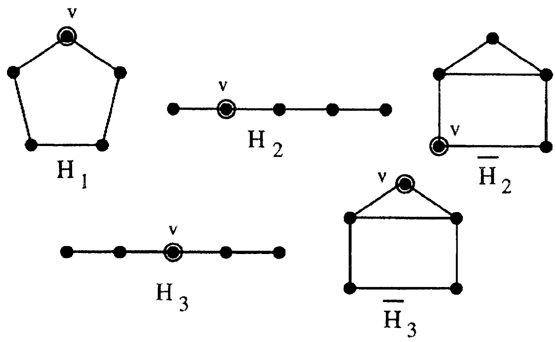

Fig. 2.

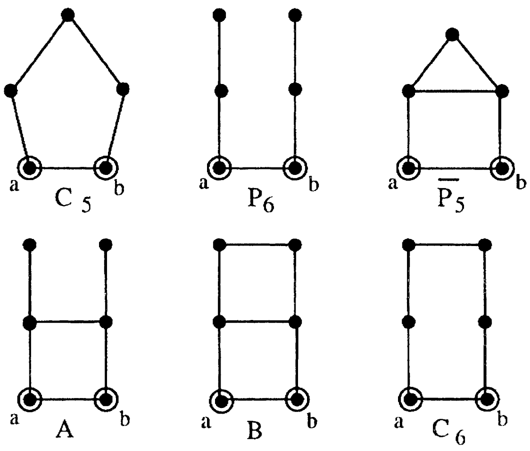

Fig. 3.

Proof. Let c, d, p and q such that both $abcd$ and $baqp$ are P_4's in G. There are two cases according as $d = p$ or $d \neq p$. If $d = p$ then cq is the only one possible edge. If $d \neq p$ then cq, cp, dq, dp are possible edges. By examining every case, it is a routine matter to obtain the announced result. □

Lemma 2. *A vertex v in a graph G is special if and only if v is $\mathscr{F}_1$-free, where $\mathscr{F}_1$ is the family of rooted graphs $\{H_1, H_2, \bar{H}_2, H_4, \bar{H}_4\}$ (see* Figs. 2 *and* 4).

Proof. By considering an end-vertex a of a symmetric wing and the rooted graphs (C_5, a), $(\bar{P}_5, a)$ and (A, a) in Fig. 3 and their (rooted) complements, we obtain rooted graphs isomorphic to $H_1, H_2, \bar{H}_2, H_4$ or to $\bar{H}_4$ (see Figs. 2 and 4). The other graphs in Fig. 3, that is (P_6, a), (C_6, a), (B, a), and their complements give rooted graphs containing a rooted graph (F, a) isomorphic either to H_2 or to $\bar{H}_2$ depicted in Fig. 2.

Conversely, let v be a vertex belonging to a rooted graph H of $\mathscr{F}_1$. If H belongs to $\{H_1, \bar{H}_2, H_4\}$ (resp. $\{H_2, \bar{H}_4\}$) then v is end-vertex in a symmetric wing in G (resp. $\bar{G}$). □

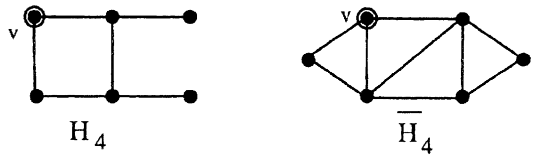

Fig. 4.

In [10] Olariu defined a graph G to be *quasi-brittle* if every induced subgraph H of G contains a special vertex, and proved that such a graph is perfectly orderable. Indeed, an attentive examination of his proof shows that he obtained the following more general result:

Theorem 1. *Let G be a graph and v be a special vertex in G. Then, G is perfectly orderable if and only if $G-v$ is perfectly orderable.*

Note that such a result was known if v is semi-simplicial [6] or if v is a charming vertex [8].

Our purpose is to generalize these previous results and to obtain new families of perfectly orderable graphs by proving statements of the following type:

Assertion A. *Let $\mathscr{F}$ be a family of rooted graphs. Let G be a graph and v be a $\mathscr{F}$-free vertex in G. Then, G is perfectly orderable if and only if $G-v$ is perfectly orderable.*

2. Definitions and notations

For terms not defined here, the reader is referred to [5]. All the graphs in this paper are simple and a graph is denoted $G=(V(G),E(G))$ (or simply $G=(V,E)$ if no confusion exists); its *complement* $\bar{G}$ is the graph $(V,\bar{E})$ where $\bar{E}=\{xy \mid x\in V,\ y\in V$ and $xy\notin E\}$. For any subset A of V, the subgraph of G (resp. $\bar{G}$) induced by the set A will be denoted by $G[A]$ (resp. $\bar{G}[A]$).

For any vertex v in V, the *neighbourhood* $N_G(v)$ of v in G is defined as the set of all vertices adjacent to v in G. These vertices are called *neighbours* of v. When there is no possibility of confusion, the notation $N(v)$ will also be used to denote the neighbourhood of v. The set of vertices in G which are not adjacent to v is denoted by $M(v)$ (this set is equal to $N_{\bar{G}}(v)$, the neighbourhood of v in $\bar{G}$). For any subset W of $V\backslash\{v\}$ such that $W\cap N(v)\neq\emptyset$ and v is not adjacent to at least one vertex in W, we will say that v *sees* W *partially*. If $W\subseteq N(v)$, we will say that v *sees* W *completely*.

If $A\subseteq V$, then $G\backslash A$ will denote the graph induced by $V\backslash A$. The particular case where $A=\{x\}$ will be denoted $G-x$; also if H is a subgraph of G, we denote by $H+x$ the subgraph induced in G by $V(H)\cup\{x\}$ (if $x\notin V(H)$).

For any path P, the *length* of P is the number of its edges. If $V(P) = \{v_1, \dots, v_k\}$ and $E(P) = \{v_i v_{i+1} \mid i \in \{1, \dots, k-1\}\}$, P is also denoted by $[v_1, \dots, v_k]$. The vertices v_1 and v_k are its *end-vertices* while any vertex v_i, with $1 < i < k$, is said to be an *internal* vertex. An induced path on k vertices will be denoted by P_k. For a P_4 with vertices a, b, c, d (in this order on the path), the notation will be slightly changed: it will be denoted $abcd$. The two internal vertices b and c will be called *midpoints* while the end-vertices a and d will be also referred as *endpoints*.

Similarly, a chordless cycle on k vertices is denoted by C_k or by $[v_1, \dots, v_k, v_1]$ if its vertex set is $\{v_1, \dots, v_k\}$ and its edge set is $\{v_i v_{i+1} \mid 1 \leqslant i \leqslant k-1\} \cup \{v_k v_1\}$.

A set $A \subseteq V$ of vertices is called a *module* if every vertex in $V \backslash A$ is either adjacent to all vertices in A, or none of them. A module in G is also a module in $\bar{G}$. A module is an *homogeneous set* if it is not a *trivial* one, i.e. not an empty set, a singleton or V itself. A graph with more than two vertices is *prime* if it only has trivial modules. Note that P_4 is the smallest prime graph and that any prime graph distinct from P_4 has at least five vertices.

Let G and H be two graphs and let v be a vertex in G. The *χ-join* of H and G into v is the graph obtained from the union of H and $G - v$ by adding all the edges xy with $x \in V(H)$ and $y \in N_G(v)$; this graph is denoted by $\chi(H, G; v)$. Since an induced P_4 in $\chi(H, G; v)$ not contained in H and not in $G - v$ has exactly one vertex in H and the three other vertices in $G - v$, note that if $v_1 < v_2 < \cdots < v_n$ is a perfect order of G and if $w_1 < w_2 < \cdots < w_p$ is a perfect order of H, then $v_1 < \cdots < v_{i-1} < w_1 < w_2 < \cdots < w_p < v_{i+1} < \cdots < v_n$ is a perfect order of $\chi(H, G; v_i)$. Then we have

Lemma 3 (Chvátal [3]). *Let G and H be two perfectly orderable graphs and let v be a vertex in G. Then $\chi(H, G; v)$ is perfectly orderable.*

3. Pleasant vertices

A vertex v in a graph G is said to be a *pleasant vertex* if v is $\mathscr{F}_0$-free, with $\mathscr{F}_0 = \{H_1, H_2, \bar{H}_2\}$ where these rooted graphs are described in Fig. 2. Note that special vertices and nice vertices are pleasant vertices. We call *pleasant* any graph in which every induced subgraph has a pleasant vertex.

For any vertex v in a graph G let $S_1(v), \dots, S_{q_v}(v)$ be the connected components of $\bar{G}[N(v)]$, and let $Q_1(v), \dots, Q_{k_v}(v)$ be the connected components of $G[M(v)]$. For the sake of simplicity we will denote these components by $S_1, \dots, S_q$ and by $Q_1, \dots, Q_k$.

We give here a structural result that we will often use.

Proposition 1. *Let v be a pleasant vertex in a graph $G = (V, E)$. For every $i \in \{1, \dots, q\}$ and every $j \in \{1, \dots, k\}$ one of the following statement is true*:

(1) *every vertex in S_i is adjacent to every vertex in Q_j*,
(2) *every vertex in S_i has no neighbour in Q_j*,

(3) *there exists a non trivial part A_i of S_i such that for every vertex x in S_i and for every vertex y in Q_j, xy belongs to E if and only if x belongs to A_i,*
(4) *there exists a non trivial part B_j of Q_j such that for every vertex x in S_i and for every vertex y in Q_j, xy belongs to E if and only if y belongs to B_j.*

Proof. If $|S_i| = 1$ or $|Q_j| = 1$ then the result is clearly true. We suppose that $|S_i| \geqslant 2$, $|Q_j| \geqslant 2$ and that none of the conditions (1)–(3) is satisfied. Then, there exist $x \in S_i$, a and $b \in Q_j$ such that $xa \in E$ and $xb \notin E$. Since Q_j is connected in G, we can suppose without loss of generality that $ab \in E$. Since S_i is connected in $\bar{G}$, there is $y \in S_i$ distinct from x such that $xy \notin E$. Since the set of vertices $\{y, v, x, a, b\}$ induces no cycle C_5, no P_5 and no $\bar{P}_5$, we have $ya \in E$ and $yb \notin E$.

Now we want to prove that any vertex h in Q_j is either adjacent to every vertex in S_i or adjacent to no vertex in S_i. A contrario, suppose that there exist h in Q_j, c and d in S_i such that $hc \in E$ and $hd \notin E$. Since S_i is connected in $\bar{G}$, we may suppose without loss of generality that $cd \notin E$. Let k be a neighbour of h in Q_j. Since the set $\{k, h, c, v, d\}$ induces no C_5, no P_5 and no $\bar{P}_5$, we have $kc \in E$ and $kd \notin E$. Since Q_j is connected, c is adjacent to every vertex in Q_j (and d is adjacent to no vertex in Q_j). But b belongs to Q_j and is adjacent to no vertex in S_i, so we obtain a contradiction. Thus, the statement 4) is true. □

Corollary 1. *Let v be a pleasant vertex in a graph $G = (V, E)$.*

(i) *Let S_i be a component of $\bar{G}[N(v)]$. Let a and b be two vertices in $M(v)$ such that $N(a) \cap S_i \neq \emptyset$, $N(b) \cap S_i \neq \emptyset$ and $N(a) \cap S_i \neq N(b) \cap S_i$. Then a and b belong to distinct components of $G[M(v)]$.*
(ii) *Let Q_j be a component of $G[M(v)]$. Let a and b be two vertices in $N(v)$ such that $N(a) \cap Q_j \neq \emptyset$, $N(b) \cap Q_j \neq \emptyset$ and $N(a) \cap Q_j \neq N(b) \cap Q_j$. Then a and b belong to distinct components of $\bar{G}[N(v)]$.*

4. Acyclic orientation and obstructions

A perfect order of G can be seen as an acyclic orientation of the edges of G such that every induced P_4, $abcd$, is not an obstruction, that is $abcd$ cannot have simultaneously (a, b) and (d, c) as arcs. Our aim is the following: given a perfect order $<$ of $G - v$, how to orient the edges incident with v and how to modify the orientation of some edges in $G - v$ so that the oriented graph G has no circuit and no obstruction. We may define a linear order $<'$ of G in the following way: let us partition $V - v$ into L and R, and denote $N(v) \cap L$ by LH, $M(v) \cap L$ by LL, $N(v) \cap R$ by RH and $M(v) \cap R$ by RL; we do not modify the orientations in $G[L]$ and $G[R]$, every edge xy such that $x \in L$ and $y \in R$ is oriented such that $x <' y$, every edge xv with $x \in L$ is oriented such that $x <' v$ and, at last, every edge yv such that $y \in R$ is oriented such that $v <' y$ (see Fig. 5). Now, our problem is to define the sets LH, LL, RH and RL in a suitable way to avoid obstructions.

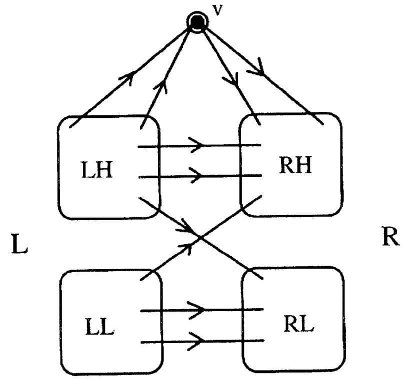

Fig. 5.

We note that, to obtain their result, Hoàng et al. [8] set $LH = N(v)$, $LL = \emptyset$, $RH = \emptyset$ and $RL = M(v)$.

We consider a pleasant vertex v of the graph G, and we suppose $G - v$ perfectly orderable. Let $<$ be a perfect order on $G - v$. Following a similar idea as used by Olariu [10], a component S_i of $\bar{G}[N(v)]$ is called *impure* if a vertex a in $M(v)$ and vertices b, d in S_i exist such that $abvd$ is a P_4 and $a < b$. A component S_i is called *pseudo-pure* if there are vertices a, b in $M(v)$ and a vertex c in S_i such that $abcv$ is a P_4 and $a < b$. A component which is neither impure nor pseudo-pure is referred to as *pure*.

Lemma 4. *No pseudo-pure component is impure.*

Proof. Assume that S_i is both pseudo-pure and impure: let a, b, x in $M(v)$ and c, y, z in S_i be such that $abcv$ and $xyvz$ and P_4 with $a < b$ and $x < y$. Since $ac \notin E$ and $bc \in E$, the vertex b sees S_i completely (by Proposition 1). As $bz \in E$, $xy \in E$ and $xz \notin E$, the component Q_j containing x is different from the component containing a and b (by Corollary 1). Then $abyx$ is an obstruction in $(G - v; <)$, a contradiction. □

We can now define the sets LH, RH, LL and RL: LH contains all the pseudo-pure components and, perhaps, some pure components; RH contains all the impure components and the remaining pure components; LL is the union of some components Q_j and RL is the union of the other components of $G[M(v)]$ (LL or RL can be empty). By construction, the oriented graph $(G; <')$ presents no circuit.

Claim 1. $(G; <')$ *presents no obstruction containing* v.

Proof. If $abcv$ is an obstruction in $(G; <')$, then a and b belong to the same component Q_j and $a < b$. Thus, c belongs to a pseudo-pure component and $v <' c$, a contradiction. If $abvd$ is an obstruction in $(G; <')$, then the component S_i containing b and d is

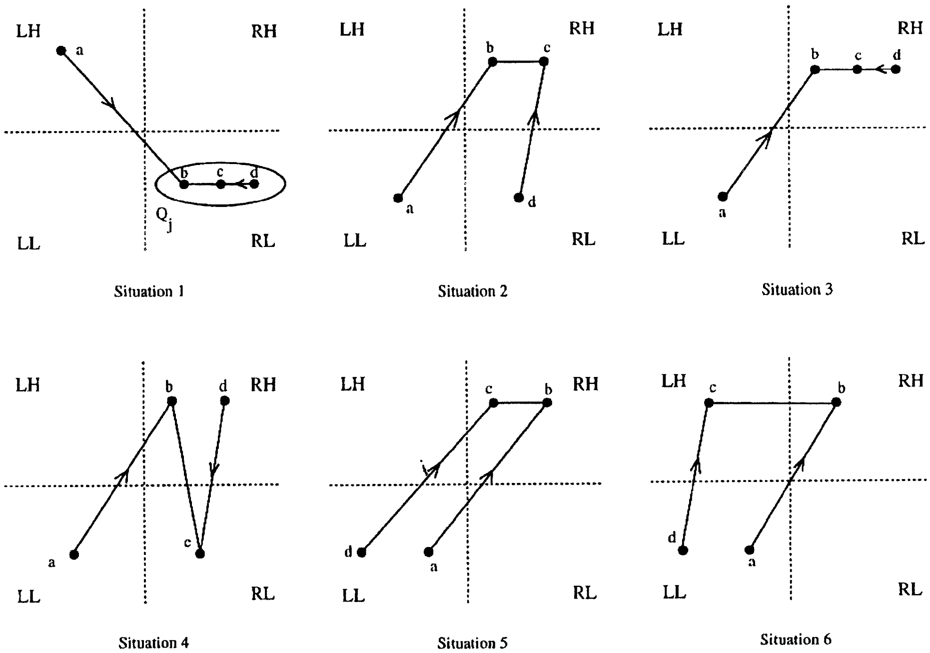

Fig. 6.

contained in LH ($d <' v$) and a is in LL ($a <' b$). Thus $a < b$ and S_i is impure, a contradiction. $\square$

We are going to observe now the possible obstructions. Let $abcd$ be an obstruction in $(G; <')$. Without loss of generality, we can suppose that $a \in L$, $b \in R$ and $b < a$. Since there is no edge between LL and RL, we notice that a or b is a neighbour of the vertex v.

Proposition 2. *If abcd is an obstruction in* $(G; <')$, *then one of the following situations depicted in* Fig. 6 *is realized*:

Situation 1: $a \in LH$ *and* $b, c, d \in Q_j \subseteq RL$.
Situation 2: $a \in LL$, $b, c \in RH$ *and* $d \in RL$.
Situation 3: $a \in LL$ *and* $b, c, d \in RH$.
Situation 4: $a \in LL$, $b, d \in RH$ *and* $c \in RL$.
Situation 5: $a, d \in LL$ *and* $c, b \in RH$.
Situation 6: $a, d \in LL$, $c \in LH$ *and* $b \in RH$.

Proof. Since $d <' c$, we notice that $c \notin L$ while $d \in R$. We distinguish three cases:
Case A: $c, d \in R$;
Case B: $d \in L$ and $c \in R$;
Case C: $c, d \in L$.
Case A: $c, d \in R$, *then* $d < c$.

*Case A*1: We suppose that $a, b \in N(v)$. Let S_i be the component (pure or pseudo-pure) containing a and let S_j be the component (pure or impure) containing b. Since $ad \notin E$ and $bd \notin E$, d belongs to $M(v)$. If $c \in N(v)$ then c belongs to S_i and this component is impure (because $d < c$), a contradiction. Thus, c and d belong to $M(v)$. But now, we consider the P_4 $dcbv$ and we see that S_j is pseudo-pure, which gives also a contradiction. Then, Case A1 does not occur.

*Case A*2: We suppose that $a \in N(v)$ and $b \notin N(v)$. As $ac, ad \notin E, b, c$ and d belong to the same component Q_j. This is the first situation.

*Case A*3. We suppose that $a \notin N(v)$ and $b \in N(v)$. As the component containing b is not pseudo-pure, one of the vertices c or d is in $N(v)$. We obtain three other possibilities: Situations 2–4.

Case B: $d \in L$ *and* $c \in R$.

*Case B*1: We suppose that $a \in N(v)$. Since $ac, bd \notin E$ and $cd \in E$, c belongs to $M(v)$ and d belongs to $N(v)$. Since $ad, bd \notin E$, a and d are in the same component S_i, b and c are in the same component Q_j. By Proposition 1, $abcd$ cannot be a P_4; this is a contradiction. Then, Case B1 cannot occur.

*Case B*2: We suppose that $a \notin N(v)$ and $b \in N(v)$. Since $bd \notin E$ and $cd \in E$, we have $d \notin N(v)$ and $c \in N(v)$. We obtain the fifth situation.

Case C: $c, d \in L$, *then* $d < c$.

*Case C*1: We suppose that $a, b \in N(v)$. As $bd \notin E$, necessarily $d \notin N(v)$. If $c \in N(v)$, then c belongs to the component S_i containing a, and S_i is impure; this is a contradiction. If $c \notin N(v)$, we consider the graph induced by $\{d, c, b\}$ and we see that the component $S_{i'}$ containing b is pseudo-pure. This is also a contradiction; then Case C1 is impossible.

*Case C*2: We suppose that $a \in N(v)$ and $b \notin N(v)$. Necessarily, a and c are in the same component S_i. If $d \in N(v)$, then d is in the component S_i and $bavd$ is a P_4 where $b < a$. Then S_i is impure, which is impossible. If $d \notin N(v)$, then $dcva$ is a P_4 and $d < c$. So the component S_i is impure. Thus, Case C2 is impossible.

*Case C*3: We suppose that $a \notin N(v)$ and $b \in N(v)$. If $c \notin N(v)$, then $dcbv$ is a P_4, where $d < c$, and the component containing b is pseudo-pure. So $c \in N(v)$, and we obtain the last situation. □

From the previous proposition, we deduce immediately the following two results (we shall see later that we can improve Corollary 3).

Corollary 2. *Let v be a pleasant vertex of a graph G. If v is endpoint of no P_5 then G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

Proof. We choose $LL = \emptyset$ and $RL = M(v)$: none of Situations 2–6 appears. Since v is endpoint of no P_5, Situation 1 cannot be realized. □

Corollary 3. *Let v be a pleasant vertex of a graph G. If v is $\{H_5, H_6\}$-free (H_5 and H_6 are described in Fig. 7), then G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

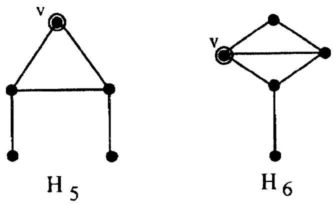

Fig. 7.

Proof. We choose $LL = M(v)$ and $RL = \emptyset$; the Situations 1, 2, 4 cannot appear. Since v is a vertex $\{H_5, H_6\}$-free, none of the Situations 3, 5, and 6 is realized. $\square$

5. Main results

Let us consider again the following

Assertion A. *Let $\mathscr{F}$ be a family of rooted graphs. Let G be a graph and v be a $\mathscr{F}$-free vertex in G. Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

We have solely to prove that if $G - v$ is perfectly orderable then G is perfectly orderable. As a consequence of Lemma 3, we have:

Claim 2. *To prove Assertion A by induction on the number of vertices, it can be supposed that G is prime.*

Proof. If every graph in $\mathscr{F}$ has p vertices or more, then this result is true for any graph G such that $|V(G)| < \min(4, p)$. Let us suppose that for any graph H such that $|V(H)| < |V(G)|$, H is perfectly orderable if w is a $\mathscr{F}$-free vertex in H and $H - w$ is perfectly orderable. We suppose that G contains a homogeneous set M.

(i) If $v \notin M$, Let H be the graph obtained from G by contracting M in a single vertex z. Since H is isomorphic to an induced subgraph of G, v is a $\mathscr{F}$-free vertex in H. Since $H - v$ is an induced subgraph of $G - v$, $H - v$ is perfectly orderable. By induction, H is perfectly orderable. The subgraph $G[M]$ is an induced subgraph of $G - v$, thus it is a perfectly orderable graph. By Lemma 3, $G = \chi(G[M], H; z)$ is perfectly orderable.

(ii) If $v \in M$ then $G[M] - v$ is an induced subgraph of $G - v$, thus $G[M] - v$ is perfectly orderable. Since v is a $\mathscr{F}$-free vertex in $G[M]$, by induction $G[M]$ is perfectly orderable. The graph $H = G - M + v$ is isomorphic to $G - M + u$ with $u \in M - v$, a subgraph of $G - v$, thus H is perfectly orderable. By Lemma 3, $G = \chi(G[M], H; v)$ is perfectly orderable. $\square$

The following statement generalizes Theorem 1.

Theorem 2. *Let G be a graph and v be a vertex $\mathscr{F}_2$-free, where $\mathscr{F}_2$ is the family $\{H_1, H_2, \bar{H}_2, \bar{H}_4\}$ (see Figs.* 2 *and* 4*). Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

Lemma 5. *Let G be a prime graph and v be a $\mathscr{F}_2$-free vertex in G. If S_i is a component of $\bar{G}[N(v)]$ and u is a non-isolated vertex in $G[M(v)]$, then u is adjacent to every vertex of S_i or none.*

Proof. On the contrary, we suppose that the vertex u sees partially the component S_i: there are vertices a, b in S_i such that $au \in E$ and $bu \notin E$. Without loss of generality, we can suppose that $ab \notin E$. As G is prime, the component Q_j containing u is not an homogeneous set, and there are vertices x in $N(v)$ and y, z in Q_j such that $xy \in E$ and $xz \notin E$. We can suppose that $yz \in E$. By Proposition 1, we have $ay \in E$, $az \in E$, $by \notin E$, $bz \notin E$, and by Corollary 1, $x \notin S_i$. Then, the rooted graph $(G[v, a, b, y, z, x], v)$ is isomorphic to $\bar{H}_4$. □

Lemma 6. *Let v be a pleasant vertex in a graph G, such that $G - v$ is perfectly orderable. Let $<$ be a perfect order of $G - v$, S_i be an impure component and x be a vertex in $M(v)$ which sees S_i completely. Then there exists a vertex $y \in S_i$ such that $x < y$.*

Proof. Let S_i be an impure component. There exist vertices $a \in M(v)$ and $b, y \in S_i$ such that $abvy$ is a P_4 and $a < b$. If $x \in M(v)$ sees S_i completely then, by Corollary 1, x and a are not in the same component of $G[M(v)]$. Since $abxy$ is not an obstruction in $(G - v; <)$, we have $x < y$. □

Proof of Theorem 2. By Claim 2, we can suppose that G is prime. We choose LH as the union of all the pure or pseudo-pure components, RH as the union of all the impure components, RL as the set of all the isolated vertices in $G[M(v)]$ and LL as the union of the other components of $G[M(v)]$.

Claim 3. *None of the Situations* 1, 3, 4 *appears.*

Proof. In Situation 1, vertices b, c, d are not isolated in $G[RL]$. In Situations 3 and 4, the vertex a belongs to $M(v)$ and sees partially the impure component containing the vertices b and d. By Lemma 5, a is isolated in $G[M(v)]$ and cannot belong to LL. □

Claim 4. *Situations* 5 *and* 6 *are not realized.*

Proof. In Situations 5 and 6, a and d are not isolated vertices in $G[M(v)]$. Let S_i (resp. S_j) be the impure component containing b (resp. c). By Lemma 5, a (resp. d)

sees completely S_i (resp. S_j). Since $ac \notin E$, S_i and S_j are distinct components. By Lemma 6, in Situation 5, there exist vertices $c', b' \in RH$ such that $dc'b'a$ is an obstruction in $(G - v; <)$; in Situation 6, there exists a vertex $b' \in RH$ such that $dcb'a$ is an obstruction in $(G - v; <)$. $\square$

A careful reading of Olariu's study about quasi-brittle graphs [10, Proof of Theorem 2, pp. 151–152] allows us to write the following lemma:

Lemma 7 (Olariu [10]). *Let v be a $\mathscr{F}_2$-free vertex in a graph G, such that $G - v$ is perfectly orderable. There exists no obstruction $abcd$ in $(G; <')$, where $a \in LL$ and $d \in RL$.*

Thus, we have

Claim 5. *Situation 2 does not appear.*

This ends the proof of Theorem 2. $\square$

Another way to generalize Theorem 1 is to consider the family $\{H_1, H_2, \bar{H}_2, H_4\}$. Unfortunately, we are unable to prove the corresponding result (we must add the rooted graph H_8 defined below; see Corollary 6).

Theorem 3. *Let G be a graph and v be a $\mathscr{F}_3$-free vertex, where $\mathscr{F}_3$ is the family of rooted graphs $\{H_1, H_2, \bar{H}_2, H_7, H_8\}$ (see Figs.* 2 *and* 8*). Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

Proof. Let $<$ be a perfect order on $G - v$. Since v is a pleasant vertex of G, by Lemma 4, no pseudo-pure component is impure. Then, let us consider the linear order $<'$ on G defined in Section 4, such that LH contains no pure component (that is LH is the union of the pseudo-pure components) and LL is empty (that is $RL = M(v)$). The oriented graph $(G; <')$ presents no circuit and, by Claim 1, no obstruction containing v. Let $A = abcd$ be an obstruction in $(G; <')$. Since $LL = \emptyset$, by Proposition 2, Situation 1 is the only one possibility. We suppose without loss of generality that a belongs to

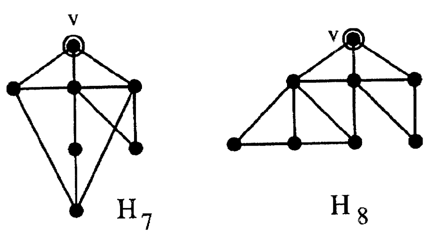

Fig. 8.

LH and b, c and d belong to RL. We denote by $C(A)$ the pseudo-pure component of $\bar{G}[N(v)]$ containing a and by $Q(A)$ the component of $G[M(v)]$ containing b, c and d. Note that, by Proposition 1, for every $x \in C(A)$ we have $bx \in E$, $cx \notin E$ and $dx \notin E$. Moreover, $a <' b$, $b < a$ and $d < c$.

Claim 6. *There exist s and t in $Q(A)$ such that $t < s$ and for all x in $C(A)$, $sx \in E$ and $tx \notin E$.*

Proof. Since $C(A) \subseteq LH$, $C(A)$ is a pseudo-pure component. Then, there exist t and s in a component Q_j such that $t < s$, $st \in E$ and for all x in $C(A)$, $sx \in E$ and $tx \notin E$. If $Q_j \neq Q(A)$ then $tsab$ is an obstruction in $(G - v; <)$, a contradiction. Thus, $Q_j = Q(A)$. $\square$

Let $\mathscr{A}$ be the set of obstructions in $(G; <')$ and let $\mathscr{C}(\mathscr{A})$ be the set of vertices x in R such that there exist $A = abcd$ in $\mathscr{A}$ and an oriented path, $P[x, b]$, from x to b, in $(G - v; <)$. Note that there is no edge xy such that $x \in \mathscr{C}(\mathscr{A})$, $y \in R \backslash \mathscr{C}(\mathscr{A})$ and $y < x$.

Now, set $L' = L \cup \mathscr{C}(\mathscr{A})$ and $R' = R \backslash \mathscr{C}(\mathscr{A})$. We apply to this new partition $\{L', R'\}$ of $V \backslash \{v\}$ the rules given in Section 3 to obtain from $(G - v; <)$ a new oriented graph $(G; <'')$.

Remark 1. If there exists xy in E such that $y <'' x$ and $y > x$, then $x \in R'$ and $y \in LH = L'H \backslash \mathscr{C}(\mathscr{A})$.

Note that, by Remark 1, Situations 2–6 do not appear in $(G - v; <'')$.

Remark 2. By definition of $\mathscr{C}(\mathscr{A})$, Situation 1 does not appear in $(G - v; <'')$.

Clearly, $(G; <'')$ presents no circuit. Remarks 1 and 2 imply that, if B is an obstruction in $(G; <'')$ then B contains v.

If $B = vv_2v_3v_4$ then $v_2 \in R'H$ (otherwise $v_2 <'' v$). Clearly, v_3 and v_4 belong to $M(v)$ and by Remark 1, $v_4 < v_3$. But the component S_i containing v_2 is pseudo-pure and then $v_2 \in LH \subseteq L'H$, a contradiction.

If $B = v_1vv_2v_3$ then $v_1 \in L'H$ (otherwise $v <'' v_1$). Clearly, v_1 and v_2 are in the same component S_i and $v_3 \in M(v)$. Since $v_3 <'' v_2$, by Remark 1, $v_3 < v_2$. Hence, S_i is an impure component, that is $v_1 \notin LH$. Since $v_1 \in L'H$, $v_1 \in \mathscr{C}(\mathscr{A})$.

Thus, there exist $A = abcd$ in $\mathscr{A}$ and an oriented path in $(G - v; <)$, $P[v_1, b]$, from v_1 to b. Since $C(A) \in LH$, $C(A) \neq S_i$, $av_1 \in E$ and $av_2 \in E$. Since $b < a$, $v_1 < a$ (otherwise $v_1P[v_1, b]bav_1$ is a circuit in $(G - v; <)$). Moreover, $v_3a \in E$, otherwise $v_1av_2v_3$ would be an obstruction in $(G - v; <)$.

We distinguish two cases:

Case A: $v_1b \in E$. Either $P[v_1, b] = v_1b$ or $v_1P[v_1, b]bv_1$ is a cycle. In each case $v_1 < b$. By Corollary 1, $v_3 \notin Q(A)$. Then $v_3b \notin E$. Note that $v_2b \notin E$, otherwise $v_1bv_2v_3$ would be an obstruction in $(G-v;<)$. By Proposition 1, v_1 is adjacent to every vertex in $Q(A)$ and v_2 is adjacent to no vertex of $Q(A)$. Now, the subgraph of G induced by $\{v,a,v_1,v_2,v_3,b,c,d\}$ is isomorphic to the rooted graph H_8 depicted in Fig. 8, a contradiction.

Case B: $v_1b\notin E$. By Claim 6 there exist s and t in $Q(A)$ such that $sa\in E$, $ta\notin E$ and $t<s$. Since v_1ast is not an obstruction in $(G-v;<)$, $v_1s\in E$ or $v_1t\in E$. Then, by Proposition 1, v_1 and v_2 have the same neighbourhood in $Q(A)$. Thus, by Corollary 1, $v_3\notin Q(A)$.

*Case B*1: *If* $v_1t\in E$. Clearly $t\neq b$ and $v_2t\in E$. Since $v_1tv_2v_3$ cannot be an obstruction, we have necessarily $t<v_1$. Moreover $tb\in E$, otherwise tv_1ab would be an obstruction in $(G-v;<)$. But now, the subgraph of G induced by $\{v,a,v_1,v_2,v_3,b,t\}$ is isomorphic to the rooted graph H_7 depicted in Fig. 8, a contradiction.

*Case B*2: *If* $v_1t\notin E$. Then necessarily, $v_1s\in E$ and $v_2s\in E$. Since tsv_2v_3 is not an obstruction in $(G-v;<)$, $v_2t\in E$. Then $v_1t\in E$, a contradiction. □

From Theorem 3, we deduce the following three corollaries:

Corollary 4. *Let G be a graph and v be a nice vertex in G* (*see Fig.* 2). *Then, G is perfectly orderable if and only if* $G-v$ *is perfectly orderable.*

Proof. The rooted graph H_3 is a rooted subgraph of H_8 and $\bar{H}_3$ is a rooted subgraph of H_7. □

Corollary 5. *Let G be a graph and v be a* $\mathscr{F}_4$*-free vertex, where* $\mathscr{F}_4$ *is the family of rooted graphs* $\{H_1,H_2,\bar{H}_2,H_5\}$ (*see Figs.* 4 *and* 7). *Then, G is perfectly orderable if and only if* $G-v$ *is perfectly orderable.*

Proof. The rooted graph H_5 is a rooted subgraph of H_7 and H_8. □

Corollary 6. *Let G be a graph and v be a* $\mathscr{F}_5$*-free vertex, where* $\mathscr{F}_5$ *is the family of rooted graphs* $\{H_1,H_2,\bar{H}_2,H_4,H_8\}$ (*see Figs.* 2, 4 *and* 7). *Then, G is perfectly orderable if and only if* $G-v$ *is perfectly orderable.*

Proof. The rooted graph H_4 is a rooted subgraph of H_7. □

Remark 3. Since H_4 is a rooted subgraph of H_7 and $\bar{H}_4$ is a rooted subgraph of H_8, we note that Theorem 1 is also a corollary of Theorem 3.

6. Conclusion

Chvátal [3] defined a graph G to be *brittle* if each induced subgraph F of G contains a vertex that is not a midpoint of any P_4 (semi-simplicial vertex) or not an endpoint of any P_4. If v is not a midpoint of any P_4 or not an endpoint of any P_4 and if $F-v$ is perfectly orderable then F is perfectly orderable. Thus, every brittle graph is perfectly orderable [3] (see also [7]).

Let $\{\mathscr{G}_i\}_{1\leqslant i\leqslant k}$ be a finite set of finite families of rooted graphs such that for any $i\in\{1,\dots,k\}$ Assertion A is true. For example, $\mathscr{G}_1=\{(P_4,v)\}$ where v denotes a midpoint of P_4, $\mathscr{G}_2=\{(P_4,v)\}$ where v denotes an endpoint of P_4, $\mathscr{G}_3$ is the family depicted in Fig. 1 ($\mathscr{G}_3$-free vertices are charming vertices), $\mathscr{G}_4$ is the family in Fig. 2 ($\mathscr{G}_4$-free vertices are nice vertices), $\mathscr{G}_5=\mathscr{F}_2$, $\mathscr{G}_6=\mathscr{F}_3$. We define a class $\mathscr{P}(\mathscr{G}_1,\dots,\mathscr{G}_k)$ of perfectly orderable graphs in the following way: a graph G belongs to $\mathscr{P}(\mathscr{G}_1,\dots,\mathscr{G}_k)$ if for any subgraph F of G there exist a family $\mathscr{G}_i$ and a $\mathscr{G}_i$-free vertex v in F. According to the previous example, we note that $\mathscr{P}(\mathscr{G}_1,\mathscr{G}_2)$ is the class of brittle graphs and that $\mathscr{P}(\mathscr{G}_1,\dots,\mathscr{G}_6)=\mathscr{P}(\mathscr{G}_3,\dots,\mathscr{G}_6)$. The last class contains brittle graphs, charming graphs, nice graphs and quasi-brittle graphs.

Although recognizing general perfectly orderable graphs is NP-complete [9], clearly, there is a polynomial time recognition algorithm for $\mathscr{P}(\mathscr{G}_1,\dots,\mathscr{G}_k)$.

One may ask the following question: is it true that if v is a pleasant vertex in a graph G such that $G-v$ is perfectly orderable, then G is perfectly orderable? In [4] we show a graph giving a negative answer to this question.

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